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Relaxation of two-temperature plasma

A. V. Bobylev,* I. F. Potapenko,* and P. H. Sakanaka†
Instituto de Física, Universidade do Estado do Rio de Janeiro, Maracanã, Rio de Janeiro 20550-013, Rio de Janeiro, Brazil

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The relaxation process of a space uniform plasma composed of electrons and one species of ions is considered. For the varied initial electron and ion temperatures, the asymptotic behavior of the solutions, under $m_e/m_i \ll 1$, are studied. Special attention has been paid to the deviation of relaxation from the classical picture, the latter being characterized by a weakly nonisothermic situation $T_e \gg (m_e/m_i)^{1/3} T_i$. An approach is developed for the detailed description of the relaxation. The perturbation of the electron distribution function, which has the character of a boundary layer for the cold electrons, is studied. The field of applicability of the well-known formulas for temperatures is extended and their corrections are obtained. The relaxation process of the two-temperature plasma is also considered numerically. A comparison of the results of the calculation with those of the asymptotic approach is made. The analytical results are confirmed by the numerical simulation results. [S1063-651X(97)02908-5]

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I. INTRODUCTION

The kinetic Landau-Fokker-Planck equation is widely used for the description of collisional plasma processes. In the absence of any losses or sources of particles as well as of energy, it describes a relaxation process to equilibrium. As an intrinsic part of physical models, both analytical and numerical, this equation has many applications in plasma physics [1–7]. That is why the relaxation problem, besides being self-important, is of interest as a test problem for many collisional plasma models [8–10]. The equalization of electron and ion temperatures is one of the oldest problems of plasma physics and was initially looked upon by Landau [11], whose solution was later refined in [12,13]. There exist a variety of papers dedicated to the solution of this problem based on kinetic equations with collision integrals of a more complicated form than the original Landau integral (see, for instance, [14]). In all these papers a standard simplified approach is widely accepted: the distribution functions are considered to be Maxwellian. Thus the problem reduces to the calculation of the second moments of the collision integral with Maxwellian distribution functions with time-dependent electron $T_e(t)$ and ion $T_i(t)$ temperatures [15]. It is clear that such an approach cannot be considered fully satisfactory for several reasons. First, it imposes a severe restriction on the initial condition $T_i \ll T_e(m_i/m_e)^{1/3}$, which can be violated in many cases of a practical interest (the magnetospheric plasma, rf ion heating, neutral beam injection in a magnetic trap, etc.). Second, the distribution functions could be far from equilibrium. In addition, the errors of the approximate differential equations for $T_e(t)$ and $T_i(t)$ remain undetermined [11–13]. There is a set of papers in which trials are made to overcome the above-mentioned difficulties through analytical (for example, the Chapmen-Enskog method) as well as by numerical methods. Unfortunately, in the majority of the cases known to us, those attempts led to wrong results.

Notice that changes of velocity directions in the isotropic problem are of no importance and normally the interaction between cold electrons and ions is supposed to be small since the energy transfer by collisions is small. This standard approach is not suitable for very slow electrons because these electrons mainly interact precisely with ions as a consequence of the known dependence of the Rutherford cross section on the relative velocity. A deeper examination of the problem in this work showed that the usual perturbation theory cannot be used since the main (by $m_e/m_i \ll 1$) part of the perturbation of the electron distribution has the character of a boundary layer in the neighborhood of the small velocities $(v \approx 0)$. In the present paper the boundary layer is thoroughly studied and corrections to the known temperature equations are obtained. Also for these equations, the region of their applicability on the initial temperatures is sensibly extended. The results allow us to have quite a clear idea of the relaxation picture for any initial temperatures.

In this problem the definitive role is being played by perturbation terms of $O(\rho)$, where, for the real electron-to-ion mass ratio, $\rho = m_e/m_i < 10^{-3}$. In numerical analyses the dissipative qualities of the usual difference schemes may substantially change the influence of these small perturbations and lead to distorted results. We should stress the important role of the completely conservative difference scheme [16,17] used here, which allows us to make calculations without numerical error accumulations, except for machine errors. The analytical calculations are also carried out on a physical level of precision, while all the results with high accuracy are confirmed by the numerical calculations. The calculations, on the other hand, showed many interesting properties of the system, which were then analyzed analytically.

In Sec. II the problem is stated and dimensionless variables are introduced. The known results of temperature relaxation as well as deviations from the classical picture are given in Sec. III. The approximate equations for $\rho \ll 1$ are

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*Permanent address: Keldysh Institute of Applied Mathematics of the Russian Academy of Sciences, 125047 Moscow, Russia.

†Also at Instituto de Física ''Gleb Watagin,'' Universidade Estadual de Campinas, CEP 13083-970, Campinas, São Paulo, Brazil.

obtained and the conditions of their applicability are considered in Sec. IV. In Secs. V and VI the distribution functions are investigated in detail and the corrected formula for the temperature equation is given. The general description of the strongly nonisotermic plasma relaxation is discussed in Sec. VII. The conclusion follows in Sec. VIII.

II. STATEMENT OF THE PROBLEM

We consider a space homogeneous plasma consisting of electrons with mass m and singly charged ions of mass M. Let N be the number of particles of each species contained in a unit volume (the electrical neutrality is improved). In this section the statement of the initial-value problem for such a system will be considered. We present kinetic equations for isotropic distribution functions in the usual Fokker-Planck form. Our aim is to describe the asymptotic behavior of the solution for the small mass ratio $\rho=m/M$. The construction of such an asymptotic solution is strongly hindered by the fact that the electron and ion masses m and M enter into the equations in a quite complicated way. For this reason, we introduce dimensionless variables and settle the dimensionless initial-value problem that has a simpler form.

The electron and ion functions, dependent only on the speed $|\vec{v}|$ and time t, will be defined by $f(v,t)$ and $F(v,t)$, respectively. The distribution functions will be considered normalized to unity, i.e.,

$$4\pi\int_0^\infty dv\ v^2 f(v,t)=4\pi\int_0^\infty dv\ v^2 F(v,t)=1, \tag{1}$$

where integrals (1) have a sense of the particle density. The system of the Landau-Fokker-Planck equations that describes the time evolution of such functions is

$$\begin{aligned}\frac{1}{\Gamma}\frac{\partial f}{\partial t}&=\frac{1}{\rho}\frac{1}{v^2}\frac{\partial}{\partial v}\left\{\frac{\hat{A}(f)+\hat{A}(F)}{3v}\frac{\partial f(v)}{\partial v}\right.\\&\qquad\left.+[\hat{B}(f)+\rho\hat{B}(F)]f(v)\right\},\\\frac{1}{\Gamma}\frac{\partial F}{\partial t}&=\rho\frac{1}{v^2}\frac{\partial}{\partial v}\left\{\frac{\hat{A}(f)+\hat{A}(F)}{3v}\frac{\partial F(v)}{\partial v}\right.\\&\qquad\left.+\left[\frac{1}{\rho}\hat{B}(f)+\hat{B}(F)\right]F(v)\right\}.\end{aligned} \tag{2}$$

Here the following notations for the integral operators $\hat{A},\hat{B}$ and the constant Γ are used:

$$\hat{A}(f)=\int_0^v dx\ x^4 f(x,t)+v^3\int_v^\infty dx\ f(x,t),$$

$$\hat{B}(f)=\int_0^v dx\ x^2 f(x,t),\quad \Gamma=\frac{16\pi^2 e^4 LN}{Mm},$$

where L is the Coulomb logarithm. It might be noticed that in a strongly nonequilibrium situation, it is necessary, strictly speaking, to use three different meanings of the Coulomb logarithm. These meanings change with time for the description of the three types of collisions ($m-m$, $M-M$, and $m-M$). Nevertheless, this circumstance has no influence on the forthcoming considerations since the final results evidently can be generalized for this case.

The nonequilibrium electron $\theta(t)$ and ion $T(t)$ temperatures, expressed in energetic units, are

$$\begin{aligned}\theta(t)&=\frac{4\pi m}{3}\int_0^\infty dx\ x^4 f(x,t),\\T(t)&=\frac{4\pi M}{3}\int_0^\infty dx\ x^4 F(x,t).\end{aligned} \tag{3}$$

We consider the initial-value problem for Eqs. (2) with the initial conditions

$$\begin{aligned}f(v,0)&=f_0(v),\quad \theta(0)=\theta_0;\\F(v,0)&=F_0(v),\quad T(0)=T_0.\end{aligned} \tag{4}$$

The energy conservation law allows us to define the plasma equilibrium temperature T_{eq} by the equality

$$\theta(t)+T(t)=\theta_0+T_0=2T_{\text{eq}}.$$

The detailed investigation of the asymptotic behavior of the solution (2) and (4) gets complicated because the small parameter ρ enters Eq. (2) in a complex manner. Therefore, it is convenient to introduce new dimensionless variables. We define

$$v_\theta^2=\frac{2\theta}{m},\quad v_T^2=\frac{2T}{M},\quad \varepsilon=\frac{v_T^2}{v_\theta^2}=\rho\,\frac{T}{\theta}, \tag{5}$$

where v_θ and v_T are the electron and ion thermal velocities, respectively, and ε is the squared ratio of their thermal speeds. Let us set

$$f(v,t)=\frac{\sqrt{\pi}}{2}\left(\frac{m}{2\pi\theta}\right)^{3/2}f'\left(\frac{v^2}{v_\theta^2},\ t'\right),$$

$$F(v,t)=\frac{\sqrt{\pi}}{2}\left(\frac{M}{2\pi T}\right)^{3/2}F'\left(\frac{v^2}{v_T^2},\ t'\right),$$

$$t=t_0t',\quad t_0=\frac{2}{\sqrt{\pi}\Gamma}\left(\frac{2\pi T_{\text{eq}}}{m}\right)^{3/2},\quad \theta=T_{\text{eq}}\theta',\quad T=T_{\text{eq}}T' \tag{6}$$

and introduce similar designations for the initial conditions (4). The electron and ion temperature equalization time t_0 is chosen as a time scale unit. It is the largest characteristic time of the problem under examination. The dimensionless temperatures $T'(t)$ and $\theta'(t)$ are selected in such a manner that $T'=\theta'=1$ in equilibrium. We substitute expressions (6) in formulas (1)–(4) and carry out the obvious but rather lengthy transformations. In the final results, we drop the prime everywhere, that is, we return to the original notations. The normalization conditions (1) can now be rewritten as

$$\int_0^\infty dx\sqrt{x}f(x,t)=\int_0^\infty dx\sqrt{x}F(x,t)=1. \tag{7}$$

As a consequence of definition (3), the following additional normalization conditions arise:

$$\int_0^\infty dx\ x^{3/2} f(x,t) = \int_0^\infty dx\ x^{3/2} F(x,t) = \frac{3}{2}. \tag{8}$$

Now the initial conditions (4) take the form

$$\begin{gathered} f(x,0)=f_0(x), \quad \theta(0)=\theta_0; \\ F(x,0)=F_0, \quad T(0)=T_0, \quad \theta_0+T_0=2. \end{gathered} \tag{9}$$

The kinetic equations for the electron and ion functions $f(x,t)$ and $F(x,t)$ can be presented in the dimensionless form

$$\begin{aligned} \theta^{3/2}\frac{\partial f}{\partial t} = \frac{1}{\sqrt{x}}\frac{\partial}{\partial x}\Bigg\{ & \frac{1}{\rho}\hat{J}(f,f) + \frac{2T}{3\theta}\frac{\partial f}{\partial x}\left[\int_0^{x/\varepsilon} dy\ y^{3/2} F(y,t) \right. \\ & \left. + \left(\frac{x}{\varepsilon}\right)^{3/2}\int_{x/\varepsilon}^\infty dy\ F(y,t)\right] + f\int_0^{x/\varepsilon} dy \sqrt{y} F(y,t) \\ & + \theta^{1/2}\theta_t x^{3/2} f(x,t)\Bigg\}, \end{aligned} \tag{10}$$

$$\begin{aligned} T^{3/2}\frac{\partial F}{\partial t} = \frac{1}{\sqrt{x}}\frac{\partial}{\partial x}\Bigg\{ & \frac{1}{\sqrt{\rho}}\hat{J}(F,F) \\ & + \frac{2}{3}\sqrt{\frac{T}{\theta}}\frac{\partial F}{\partial x}\left[\varepsilon\int_0^x dy\ y^{3/2} f(\varepsilon y,t) \right. \\ & \left. + x^{3/2}\int_{x\varepsilon}^\infty dy\ f(y,t)\right] + \left(\frac{T}{\theta}\right)^{3/2} F\int_0^x dy\sqrt{y} f(\varepsilon y,t) \\ & + T^{1/2} T_t x^{3/2} F(x,t)\Bigg\}, \end{aligned} \tag{11}$$

where electron-electron collisions are described by the operator

$$\begin{aligned} \hat{J}(f,f) = \frac{2}{3}\frac{\partial f}{\partial x}\left[\int_0^x dy\ y^{3/2} f(y,t) + x^{3/2}\int_x^\infty dy\ f(y,t)\right] \\ + f\int_0^x dy\sqrt{y} f(y,t) \end{aligned} \tag{12}$$

and ion-ion collisions are described by a similar operator $\hat{J}(F,F)$. On the right-hand side of the kinetic equations (10) and (11) the first term describes the collisions of one sort of particles, next are terms that describe the collisions of different species, and finally the last term is related to the variation of the scaling (the temperature) in time.

The dimensionless temperatures $\theta(t)$ and $T(t)$ now enter explicitly into the system (10) and (11). They are defined as solutions of the evolutionary equations that appear as a consequence of the compatibility of Eqs. (10) and (11) with requirements (8) and look like

$$\begin{aligned} & \theta + T = 2, \\ & \theta^{3/2}\theta_t = \tfrac{2}{3}\int_0^\infty dx\ F(x)\left\{T\int_0^x dy\sqrt{y} f(\varepsilon y,t) \right. \\ & \qquad \left. - \theta\sqrt{x}\int_{\varepsilon x}^\infty dy\ f(y)\right\}. \end{aligned} \tag{13}$$

It is worth commenting on the physical meaning of variable x. It represents the relative energy of particles, where the scaling unit is the temperature of this type of particle. Obviously, the scale varies with time. It is useful to have in mind that the integral

$$\int_0^{x_0} dy\sqrt{y} f(y) = P(v \leqslant \sqrt{x_0} v_\theta)$$

is quantitatively equal to the probability of the electron having a velocity v that is not larger than $\sqrt{x_0} v_\theta$.

Equations (9)–(13) completely define the statement of the Cauchy problem in the dimensionless form. It is assumed that the initial conditions (9) satisfy the normalization requirements (7) and (8). Under those circumstances the satisfaction of Eqs. (7) and (8) for all $t>0$ is a consequence of Eqs. (10)–(13). Now we shall study the asymptotic behavior of the system (9)–(13) for small mass ratio ρ.

III. LIMITING SOLUTIONS

Here we study the limiting solutions of Eqs. (9)–(13), as $\rho \to 0$. In fact, the dimensionless equations are substantially easier than the original ones since all the quantities to be included in Eqs. (9)–(13) are of order one, except two small parameters ρ and $\varepsilon = \rho T/\theta$ [Eq. (5)]. In Sec. III we present the classical results on temperature relaxation. The relaxation picture goes like this. During the electron-electron collisional time τ_m the electrons become distributed according to a Maxwell function. After that the ion Maxwellization occurs over the ion-ion collisional time period $\tau_M \gg \tau_m$. Next the temperature equalization goes over the time interval $\tau_{\theta_0} \gg \tau_M \gg \tau_m$. The electron and ion temperature evolution is described by a well-known equation [13]. We discuss the applicability of this solution. Such a relaxation picture is based on the following restrictions: the mass ratio is small $\rho \ll 1$ and the initial temperatures are very near $\theta \gg \rho^{1/3} T$. We briefly consider the cases of relatively hot electrons $\theta_0 > T_0$ and relatively cold electrons $T_0 > \theta_0$ under which the latter restriction can be violated.

Let us fix the initial conditions (9) and consider the formal limits of the solution of the Cauchy problem (10)–(13) when $\rho \to 0$ and when some additional limitations on the independent variable t (time) are imposed. After a trivial analysis we obtain three known limiting solutions. Each one corresponds to their type of limiting transition. Below these solutions are considered and the processes that they indicate are described.

(i) Relaxation of the electron distribution function. Let $\rho \to 0$ and $t \to 0$; t/ρ is finite. Then

$$\begin{gathered}\theta\to\theta_0,\quad T\to T_0;\\ f\to f_1\left(x,\ \frac{t}{\rho\theta^{3/2}}\right),\quad F\to F_0(x),\end{gathered}\tag{14}$$

where $f_1(x,\tau)$ is the Cauchy problem solution for the Landau-Fokker-Planck equation

$$\frac{\partial f}{\partial\tau}=\frac{1}{\sqrt{x}}\frac{\partial}{\partial x}\hat{J}(f_1,f_1),\quad f_1(x,0)=f_0(x).$$

(ii) Relaxation of the ion distribution function. Let $\rho\to 0$ and $t\to 0$; $t/\sqrt{\rho}$ is finite. Then

$$\begin{gathered}\theta\to\theta_0,\quad T\to T_0;\\ f\to\frac{2}{\sqrt{\pi}}e^{-x},\quad F\to F_1\left(x,\ \frac{t}{\sqrt{\rho}T^{3/2}}\right),\end{gathered}\tag{15}$$

where $F_1(x,\tau)$ is the Cauchy problem solution for the equation

$$\frac{\partial F}{\partial\tau}=\frac{1}{\sqrt{x}}\frac{\partial}{\partial x}\hat{J}(F_1,F_1),\quad F_1(x,0)=F_0(x).$$

(iii) Relaxation of temperatures. Let $\rho\to 0$; t is finite. Then

$$\begin{gathered}\theta\to\theta_0,\quad T\to T_0;\\ f\to\frac{2}{\sqrt{\pi}}e^{-x},\quad F\to\frac{2}{\sqrt{\pi}}e^{-x},\end{gathered}\tag{16}$$

where $\theta(t)$ and $T(t)$ are defined from the system

$$\begin{gathered}\theta+T=2,\quad \theta^{3/2}\frac{d\theta}{dt}=\frac{4}{3\sqrt{\pi}}(T-\theta);\\ \theta(0)=\theta_0,\quad T(0)=T_0.\end{gathered}\tag{17}$$

The combination of those three solutions (14)–(16) gives the known presentation of the general solution structure of the problem (9)–(13) with the small finite parameter $\rho\ll 1$. This solution is composed of the external limiting solution (16) for the time interval $t\geqslant 1$ and two internal limiting solutions (15) and (14) for time intervals $t\sim\sqrt{\rho}$ and $t\sim\rho$, respectively, which link the external solution with the initial conditions. Such a picture is described in the Landau paper [11] and in what follows we will be interested in its deviations as well as in clarifying its details.

In order to understand when those deviations are possible, we introduce the characteristic times of the three physical processes that have already been considered:

$$\tau_m=\rho\theta_0^{3/2},\quad \tau_M=\sqrt{\rho}T_0^{3/2},\quad \tau_{\theta_0}=\theta_0^{3/2}.\tag{18}$$

In the definition of τ_{θ_0} for simplicity, we have dropped the numerical multipliers of order one [compare with Eq. (17)]. For better visualization, we show these values graphically in a representative time scale [Fig. 1(a)].

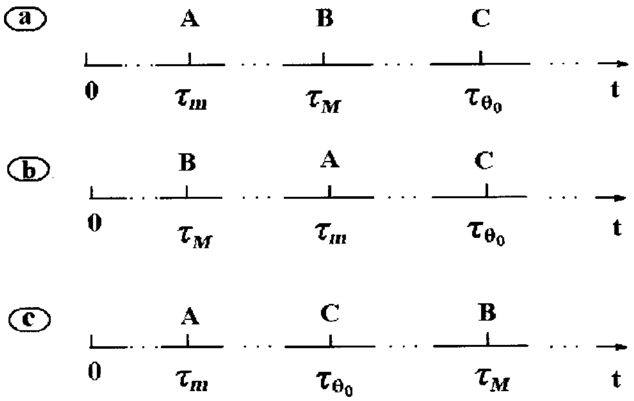

FIG. 1. (a) Classical case of the temperature relaxation: $\tau_m\simeq\rho\theta_0{}^{3/2}$, $\tau_M\simeq\sqrt{\rho}T_0{}^{3/2}$, and $\tau_{\theta_0}\simeq\theta_0{}^{3/2}$. (b) Case of the relatively hot electron relaxation: $\tau_M\simeq\sqrt{\rho}T_0{}^{3/2}$, $\tau_m\simeq 2^{3/2}\rho$, and $\tau_{\theta_0}\simeq 2^{3/2}$. (c) Case of the relatively cold electron relaxation: $\tau_m\simeq 2^{3/2}\rho$, $\tau_{\theta_0}\simeq 2^{3/2}$, and $\tau_M\simeq 2^{3/2}\rho^{1/2}$.

The limiting solution (14) describes apparently a certain ‘‘neighborhood’’ of point A or, in other words, is applicable for the time interval less than the ion-ion collisional time $0\leqslant t\leqslant\tau_M$. Similarly, Eq. (15) describes a neighborhood of point B or is applicable for the time interval less than the temperature equalization time $\tau_m\leqslant t\leqslant\tau_{\theta_0}$. Finally, the external equations describe the neighborhood (opened on the right-hand side) of point C; therefore, they are applicable for $t\gg\tau_M$. Since it was implicitly assumed above, the quantities θ_0 and T_0 have no connection with the small parameter ρ, thence $\tau_m\ll\tau_M\ll\tau_{\theta_0}$ for $\rho\ll 1$. Such a picture evidently supposes that the relative distribution of points A, B, and C coincides with the representation of Fig. 1(a). This time inequality is equivalent to the well-known restrictions on the initial temperatures

$$\rho^{1/3}\ll\frac{T_0}{\theta_0}\ll\rho^{-1/3}.\tag{19}$$

The inequality (19) is imposed on the initial temperatures T_0 and θ_0 in order permit one to make available the applicability of the composed solution (14)–(17). The appearance of such a condition is evident since the limiting solutions (14)–(17) were obtained by passing to the limit $\rho\to 0$ with the fixed initial condition (9) and in this case the inequality (19) is automatically satisfied. From the mathematical point of view, the violation of Eq. (19) may be bound with a conditional, relatively to the initial conditions (9), limiting transitions. At the same time, the inequality (19) can be violated in many cases of practical interest for the real mass ratio $\rho^{1/3}\leqslant 0.1$. Let us briefly discuss situations that lead to the violation of Eq. (19) due to the difference in the initial temperatures. It is worth having in mind that for the problem under consideration [Eqs. (9)–(13)] $T_0+\theta_0=2$.

Two basically different situations arise. The first is *relatively hot electrons*: $\theta_0\geqslant T_0\rho^{-1/3}$. In our problem such a situation can be described by decreasing T_0, which leads to the automatic increasing of θ_0. Graphically, point B in Fig. 1(a) moves to the left- and points A and C move to the right-hand side. The intersection of the trajectories of points A and B happens at the temperatures $\theta_0\simeq 2$, $T_0\simeq 2\rho^{1/3}$. By a further decrease of T_0, points A and C practically do not move but point B keeps moving to the left-hand side [Fig. 1(b)]. Therefore, the characteristic intervals τ_m and τ_M of the internal limiting solutions may become comparable. Under definite conditions ($\theta_0 \gg T_0 \rho^{-1/3}$), the processes of the electron and ion relaxation can even change locations in time, but it will have no influence on the external solution. As before, the full solution consists of three parts without any interaction between each other. The electron and ion relaxation processes have no links. For the external solution (the temperature relaxation), it is only important that both of the distributions are Maxwellian.

The second situation is *relatively cold electrons*: $\theta_0 \approx \rho^{1/3} T_0$. To obtain this inequality it is necessary to decrease θ_0 and simultaneously to increase T_0. Points A and C in Fig. 1(a) move to the left-hand side and point B moves to the right-hand side. At temperatures $\theta_0 \simeq 2\rho^{1/3}$, $T_0 \simeq 2$, points B and C coincide. As before, the first relaxation stage is described by Eq. (14) for electrons. After that the picture becomes more complicated since $\tau_M \gg \tau_{\theta_0}$ [see Fig. 1(c)]. At first sight, the temperature equation (17) becomes contradictory. It was obtained under the assumption that the distribution functions are Maxwell functions, but there are no reasons to make this supposition. The standard approach is valid under the restriction $\theta_0 \gg \rho^{1/3} T_0$, which is violated in the case of the relatively cold electrons. Thus this case needs to be specifically examined.

IV. APPROXIMATE EQUATIONS AND CONDITIONS OF THEIR APPLICABILITY

As it was considered above, the deviation from the solution (14)–(17) can be caused first by the difference between the initial temperatures and second by a very specific mechanism of the electron-ion interaction. In this section we pass to the detailed investigation of the system (9)–(13) and replace it by approximate equations for the small mass ratio $\rho \ll 1$. After that we discuss the applicability condition of the approximate system and the processes that lie beyond the approximation limits. The analysis shows that very cold electrons have a non-Maxwellian distribution. For the electron distribution the presence of a composed boundary layer in the neighborhood of $0 < x \leqslant \varepsilon^{2/3}$ is established. The correct boundary condition for the electron distribution function at $x=0$ is obtained. Below we show that the temperature equation (17) is valid under the slower restriction on the initial temperatures $\varepsilon \ll 1$ at $t=0$, but not the restriction (19). Details are presented in what follows.

On the right-hand side of Eqs. (10)–(13) we keep the singular terms at $\rho=0$ and the remaining terms will be replaced by its limit values ($\rho \to 0$ and x, T, θ are fixed). As a result, using the normalization (7) and (8), we obtain the approximate kinetic equations

$$\theta^{3/2} \frac{\partial f}{\partial t} = \frac{1}{\sqrt{x}} \frac{\partial}{\partial x} \left\{ \frac{1}{\rho} \hat{J}(f,f) + \left(\frac{T}{\theta} \frac{\partial f}{\partial x} + f \right) + \theta^{1/2} \frac{\partial \theta}{\partial t} x^{3/2} f \right\}, \tag{20}$$

$$T^{3/2} \frac{\partial F}{\partial t} = \frac{1}{\sqrt{x}} \frac{\partial}{\partial x} \left\{ \frac{1}{\sqrt{\rho}} \hat{J}(F,F) + \frac{2}{3} \left(x \frac{T}{\theta} \right)^{3/2} \left[\frac{\theta}{T} \frac{\partial F}{\partial x} \int_0^\infty dy\, f(y) + F f(0) \right] + \sqrt{T} \frac{\partial T}{\partial t} x^{3/2} F \right\}, \tag{21}$$

$$\theta^{3/2} \frac{\partial \theta}{\partial t} = \frac{2}{3} \left[T f(0) - \theta \int_0^\infty dx\, f(x) \right], \tag{22}$$

where $\hat{J}(f,f), \hat{J}(F,F)$ are defined in Eq. (12). It is necessary to check the compatibility condition of Eqs. (20)–(22) with requirements (7) and (8), which have already been used for their deduction. Multiplying Eqs. (20) and (21) by the factor $x^{1/2}$ and by the factor $x^{3/2}$ and integrating, we obtain the additional compatibility condition in the form of a boundary condition for the electron distribution function $f(x,t)$,

$$\left(\frac{T}{\theta} \frac{\partial f}{\partial x} + f \right)_{x=0} = 0. \tag{23}$$

The physical meaning of condition (23) will be discussed below. Thus we arrived at the simplified system (20)–(23) starting from the system of equations (10)–(13).

Let us consider what has been assumed for such a simplification. Let us formulate the assumptions in a standard form of enhanced inequalities

$$\rho \ll 1, \tag{24a}$$

$$\varepsilon = \rho \frac{T}{\theta} \ll 1, \tag{24b}$$

$$x \gg \varepsilon, \tag{24c}$$

$$x \ll \varepsilon^{-1}. \tag{24d}$$

Condition (24a) is evidently always fulfilled. Condition (24b), in fact, is reduced to the indication of the lower boundary of the initial electron temperature $\theta_0 \gg 2\rho$. The processes that occur by the violation of this condition (strong nonisotermic plasma) will be analyzed later in Sec. VII. Condition (24d) is used only for the derivation of Eq. (21) and shows that it is inapplicable in the far tail region $x \sim \varepsilon^{-1}$ of the ion distribution. When $\varepsilon \ll 1$ this regions contains a negligible amount of ions, whence this region presents no practical interest. Therefore, condition (24d) may be considered as always being satisfied.

The most interesting is inequality (24c), which is used only for the derivation of Eq. (20). In reality, on the one hand, inequality (24c) shows that Eq. (20) is inapplicable to the description of electrons with velocities on the order of the thermal ion velocity $v \sim v_T$ (5). This inequality is not burdensome since the number of such electrons is very small. On the other hand, it is intuitively intelligible that precisely condition (24c) is related to the appearance of the boundary condition (23), which is not satisfied by the limiting, at $\rho=0$, solutions (15) and (16). Therefore, we consider the slow electrons in more detail, returning to the exact equation (10), which we rewrite for convenience as

$$\theta^{3/2}\frac{\partial f}{\partial t}=\frac{1}{\rho}\frac{1}{\sqrt{x}}\frac{\partial}{\partial x}\left\{[D_m(x)+D_M(x)]\frac{\partial f}{\partial x}+[R_m(x)+R_M(x)]f+\rho\sqrt{\theta}\frac{\partial\theta}{\partial t}x^{3/2}f\right\}. \quad (25)$$

Here the coefficients D_m, R_m correspond to the electron-electron collisions

$$D_m(x)=\frac{2}{3}\left[\int_0^x dy\ y^{3/2}f(y)+x^{3/2}\int_x^\infty dy\ f(y)\right],$$
$$R_m(x)=\int_0^x dy\ y^{1/2}f(y) \quad (26)$$

and take a simple form in the region $x\ll 1$,

$$D_m(x)\simeq\tfrac{2}{3}x^{3/2}\int_0^\infty dy\ f(y),\ R_m(x)\simeq\tfrac{2}{3}x^{3/2}f(0),\quad x\ll 1.$$

The coefficients D_M, R_M,

$$D_M(x)=\frac{2}{3}\,\varepsilon\left[\int_0^{x/\varepsilon}dy\ y^{3/2}F(y)+\left(\frac{x}{\varepsilon}\right)^{3/2}\int_{x/\varepsilon}^\infty dy\ F(y)\right], \quad (27)$$
$$R_M(x)=\rho\int_0^{x\varepsilon}dy\ y^{1/2}F(y)$$

correspond to the electron-ion collisions.

For fixed $\rho\ll 1$ (and $\varepsilon\ll 1$) and for different magnitudes of the independent variable x, the comparison by the order of coefficients (26) and (27) yields the result

(i)	$x\sim\varepsilon$	$D_m\sim\varepsilon^{3/2}$,	$R_m\sim\varepsilon^{3/2}$	$D_M\sim\varepsilon$,	$R_M\sim\rho$	Eq. (20) is inapplicable
(ii)	$\varepsilon\ll x\ll\varepsilon^{2/3}$	$D_m\ll\varepsilon$,	$R_m\ll\varepsilon$	$D_M\sim\varepsilon$,	$R_M\sim\rho$	Eq. (20) is applicable
(iii)	$x\sim\varepsilon^{2/3}$	$D_m\sim\varepsilon$,	$R_m\sim\varepsilon$	$D_M\sim\varepsilon$,	$R_M\sim\rho$	Eq. (20) is applicable
(iv)	$x\gg\varepsilon^{2/3}$	$D_m\gg\varepsilon$,	$R_m\gg\varepsilon$	$D_M\sim\varepsilon$,	$R_M\sim\rho$	Eq. (20) is applicable.

From the above table it is clear that the electron-ion collision coefficients D_M, R_M have a quite specific influence. In the domain around $x\sim 1$ the perturbation is very small, but in the vicinity of $0\leqslant x\leqslant\varepsilon^{2/3}$ this perturbation enhances sensibly. In such a case one says that in the vicinity of the point $x\simeq 0$ there exists a boundary layer with a width of $\delta\sim\varepsilon^{2/3}$. Within its internal domain $0\leqslant x<\varepsilon$ the coefficients D_M, R_M depend essentially on the ion distribution $F(x,t)$. For $x\gg\varepsilon$ this dependence disappears and the solution satisfies Eq. (20). Therefore, the main part (ii) and (iii) of the boundary layer $\varepsilon\ll x\leqslant\varepsilon^{2/3}$ is correctly described by Eq. (20), which will be investigated in detail in what follows. For the construction of the solution in the internal domain (i) it is necessary to use the exact formulas (27) for the coefficients D_M, R_M. Let us briefly describe the behavior of the solution in this domain.

We maintain in Eq. (25) only the main (under $0<x\leqslant\varepsilon$) terms that correspond to the electron-ion interaction. Then we set $z=x/\varepsilon$ and $f(x,t)=\overline{f}(z,t)$; as a result we have

$$\rho^{5/2}T^{3/2}\frac{\partial\overline{f}}{\partial t}\simeq\frac{1}{\sqrt{z}}\frac{\partial}{\partial z}\left\{\frac{2}{3}\left[\int_0^z dy\ y^{3/2}F(y)+z^{3/2}\int_z^\infty dy\ f(y)\right]+\rho\overline{f}\int_0^x dy\ y^{1/2}F(y)\right\}. \quad (28)$$

This equation describes a very fast process with the characteristic time interval $\Delta\tau\sim\rho(\rho T_0)^{3/2}$, such that the time dependence of functions $T(t)$ and $F(x,t)$ may be neglected. With the increasing t, the function $f(z,t)$ rapidly (with the same characteristic time $\Delta\tau$) approaches the stationary solution of Eq. (29),

$$\overline{f(z)}=\overline{f(0)}\exp\left[-\rho\int_0^z ds\ \frac{R(s)}{D(s)}\right].$$

Here $D(x)$ and $R(x)$ are defined by Eq. (26),

$$D(x)=\frac{2}{3}\left[\int_0^x dy\ y^{3/2}F(y)+x^{3/2}\int_x^\infty dy\ F(y)\right],$$

$$R(x)=\int_0^x dy\ y^{1/2}F(y)$$

and $\overline{f(0)}$ has to be defined by joining the solution in the region $x\sim\varepsilon$ with the solution in the region $x\gg\varepsilon$. In fact, $f(0)$ should be substituted by $f(0,t)$, which is a ''slowly'' changing function in comparison to the characteristic time $\Delta\tau$. Finally returning to the variable x, in the region (i) we get the electron distribution function

$$\overline{f(x,t)}=\overline{f(0)}\exp\left[-\rho\int_0^{x/\varepsilon}ds\ \frac{R(s)}{D(s)}\right],\quad 0\leqslant x\leqslant\varepsilon.$$

For $x\gg\varepsilon$, i.e., in the region of applicability of Eq. (20) this function has the form

$$\overline{f(x,t)}\simeq\overline{f(0)}\exp\left\{-\frac{\theta}{T}x-\rho\int_0^\infty ds\left[\frac{R(s)}{D(s)}-1\right]\right\}$$

or, taking into account that we are dealing with $x\ll 1$ and $\rho\ll 1$, one obtains

$$\overline{f(x,t)}\simeq\overline{f(0,t)}\exp\left\{1-\rho\int_0^\infty ds\left[\frac{R(s)}{D(s)}-1\right]\right\}\left(1-\frac{\theta}{T}x\right). \tag{29}$$

On the other side, for $x\rightarrow 0$, the solution of Eq. (20) can be linearized,

$$f(x,t)\simeq f(0,t)+xf_x(0,t),\quad x\rightarrow 0. \tag{30}$$

From the comparison of Eqs. (29) and (30) it follows that the conditions for the joining of the solution $f(x,t)$ of Eq. (20) with the solution $\overline{f(x,t)}$ in the internal domain (i) can be written in the form

$$f(0,t)=\overline{f(0,t)}\left\{1-\rho\int_0^\infty ds\left[\frac{R(s)}{D(s)}-1\right]\right\},$$
$$f(0,t)+\frac{T}{\theta}f'_x(0,t)=0. \tag{31}$$

The second equality coincides with the boundary condition (23). Thence we explained the origin of the condition (23), as well as uniquely defined the solution in the internal domain $0\leqslant x\leqslant\varepsilon$ supposing that the function $F(x,t)$ and the solution of Eq. (20) are known. It is important that this solution deviates slightly from the straight line (30), only on of $O(\rho)$. Moreover, the error has a higher order of smallness if ions are distributed by a Maxwellian [under this circumstance the integral in Eq. (31) is equal to zero]. In fact, for the nonstationary problem the boundary condition (23) is established over the extremely fast time period $\Delta\tau\sim\rho(\rho T_0)^{3/2}$. Therefore, below we shall assume that the initial conditions (9) satisfy this boundary condition.

The remarkable property of the system (20)–(23) should be stressed. It consists in the following. The electron distribution function $f(x,t)$ and the temperatures T,θ by no means depend on the ion function $F(x,t)$. Hence, if we have no interest in the function $F(x,t)$ itself, Eq. (21) can simply be dropped. On the other hand, the limiting under ($\rho\ll 1$ and fixed t) the solution of Eq. (20) can only be a Maxwell distribution (16) for $f(x,t)$, which after the substitution into Eq. (22) leads to the known equation (17) for the temperatures. The obtained equations are the same, but now the condition

$$\varepsilon=\rho\frac{T}{\theta}\ll 1 \tag{32}$$

is the unique requirement. On account of temperature equilibration, it is sufficient that Eq. (32) be satisfied at $t=0$, that is, the condition (32) can be considered as a limitation on the initial temperatures. This limitation is weaker than the one that follows from the condition (19): $\varepsilon_0\ll 10^{-2}$.

Summarizing, in this section the approximate equations (20)–(23) were obtained and its applicability conditions (24) were discussed. The presence of a composed boundary layer (i)–(iii) for the electron distribution was established. The investigation of the function $\overline{f(x,t)}$ in the internal boundary layer domain (i) has shown that this solution can be substituted with a precision of $O(\rho)$ by the boundary condition (23) for Eq. (20). The analysis has shown that the unique, except for $\rho\ll 1$, essential applicability condition of the temperature equation (17) and approximate equations (20)–(23) is a weak inequality (32) ($\varepsilon_0\ll 1$), but not Eq. (19).

V. ELECTRON DISTRIBUTION AND TEMPERATURE EQUATION

Here, using the obtained approximate equations ($\varepsilon\ll 1$), we find the adjusted electron distribution function in a cold velocity region. Simultaneously, an adjustment is used for the temperature equation that leads to the corrected formula for temperatures. Now we pass to the examination of the initial-value problem for the electron distribution $f(x,t)$. For convenience, we present once more the closed system of equations for the electron function (20),

$$\theta^{3/2}\frac{\partial f}{\partial t}=\frac{1}{\sqrt{x}}\frac{\partial}{\partial x}\left\{\frac{1}{\rho}\hat{J}(f,f)+\left(\frac{T}{\theta}\frac{\partial f}{\partial x}+f\right)+\theta^{1/2}\frac{\partial\theta}{\partial t}x^{3/2}f\right\}, \tag{33}$$

$$\hat{J}(f,f)=\frac{2}{3}\frac{\partial f}{\partial x}\left[\int_0^x dy\,y^{3/2}f(y,t)+x^{3/2}\int_x^\infty dy\,f(y,t)\right]+f\int_0^x dy\,\sqrt{y}f(y,t), \tag{34}$$

with the boundary condition (23), the initial condition (9),

$$\left(T\frac{\partial f}{\partial x}+\theta f\right)_{x=0}=0,\quad f(x,0)=f_0(x),\quad \theta(0)=\theta_0,$$
$$T(0)=T_0+2-\theta_0, \tag{35}$$

the normalization conditions (7) and (8),

$$\int_0^\infty dy\,\sqrt{y}f(y,t)=\frac{2}{3}\int_0^\infty dy\,y^{3/2}f(y,t)=1, \tag{36}$$

and for temperatures $\theta(t)$ and $T(t)$ (22)

$$\theta+T=2,\quad \theta^{3/2}\frac{\partial\theta}{\partial t}=\frac{2}{3}\left[Tf(0)-\theta\int_0^\infty dx\,f(x)\right]. \tag{37}$$

From Eqs. (33), (34), and (37) for $\rho=0$, we formally obtain the solution

$$f^{(0)}(x)=\frac{2}{\sqrt{\pi}}e^{-x},\quad \theta^{3/2}\frac{d\theta}{dt}=\frac{4}{3\sqrt{\pi}}(T-\theta). \tag{38}$$

As can be seen, this solution does not satisfy the boundary condition at $x=0$. It is clear that $f^{(0)}(x)$ is the external (for $x\gg\varepsilon^{2/3}$) solution and should be corrected in the vicinity of $x=0$. To adjust Eq. (38), we let, in Eq. (33),

$$f(x,t)=\frac{2}{\sqrt{\pi}}e^{-x}[1+\varphi(x,t)], \tag{39}$$

where $\varphi(x,t)$ is a small additional function, and neglect the squared terms of $\varphi(x,t)$ in $\hat{J}(f,f)$. The result is

$$\theta^{3/2}e^{-x}\,\frac{\partial\varphi}{\partial t}=\frac{1}{\sqrt{x}}\,\frac{\partial}{\partial x}\,e^{-x}\left\{\frac{2}{\rho\sqrt{\pi}}\,\hat{L}f+\frac{T}{\theta}\,\frac{\partial\varphi}{\partial x}+\left[\left(1-\frac{T}{\theta}\right)+\theta^{1/2}\,\frac{d\theta}{dx}\,x^{3/2}\right](1+\varphi)\right\}. \tag{40}$$

Here $\hat{L}$ is a linear operator defined by

$$\hat{L}\varphi=\int_0^\infty dy\; e^{-y}K(x,y)\left[\frac{\partial\varphi(x)}{\partial x}-\frac{\partial\varphi(y)}{\partial y}\right], \tag{41}$$

$$K(x,y)=\tfrac{2}{3}\min(x^{3/2},y^{3/2}).$$

The boundary condition (35) for $x=0$ and conditions (36) now take the form

$$[T\varphi_x+(\theta-T)(1+\varphi)]_{x=0}=0,$$

$$\int_0^\infty dy\; e^{-y}y^{1/2}\varphi(y)=\int_0^\infty dy\; e^{-y}y^{3/2}\varphi(y)=0. \tag{42}$$

We suppose that $\varphi\to 0$ (i.e., $f\to f^0$) for $\rho\to 0$. However, to satisfy the boundary condition the derivative $\varphi_x(0,t)$ has to be finite in the limit as $\rho=0$. Taking such a behavior of $\varphi(x,t)$ into account, we neglect in Eqs. (40) and (42) the function $\varphi(x,t)$, in comparison with unity, in those components that are proportional to $(1+\varphi)$; but the components that are proportional to $\varphi_x(x,t)$ will be kept unchanged. Accordingly, we obtain

$$\theta^{3/2}e^{-x}\,\frac{\partial\varphi}{\partial t}=\frac{1}{\rho}\,\frac{1}{\sqrt{x}}\,\frac{\partial}{\partial x}\,e^{-x}\left[\frac{2}{\rho\sqrt{\pi}}\,\hat{L}\varphi+\varepsilon\,\frac{\partial\varphi}{\partial x}+\left(\rho-\varepsilon+\rho\theta^{1/2}\,\frac{d\theta}{dt}\,x^{3/2}\right)\right]; \tag{43}$$

$$\varepsilon\varphi_x+(\rho-\varepsilon)=0, \tag{44}$$

$$\int_0^\infty dy\; e^{-y}y^{1/2}\varphi(y)=\int_0^\infty dy\; e^{-y}y^{3/2}\varphi(y)=0.$$

From the consistency of Eq. (43) with conditions (44), the simplified temperatures' equations follow

$$\theta^{3/2}\,\frac{d\theta}{dt}=\frac{4}{3\sqrt{\pi}}\left[T-\theta-T\int_0^\infty dy\; e^{-y}\varphi_y(y,t)\right],\quad T=2-\theta, \tag{45}$$

instead of Eqs. (35). Over the characteristic time period $\rho^{-1}\gg 1$ the solution of Eq. (43) approaches the quasistationary solution $\varphi[x;\varepsilon(t)]$, which makes the right-hand side vanish, and depends on time only through the slowly changing function $\varepsilon(t)$. To find this solution it is convenient to take in Eq. (43)

$$\rho\theta^{1/2}\,\frac{d\theta}{dt}=\frac{4}{3\pi}\,\lambda(\varepsilon)(\varepsilon-\rho),$$

where the symbol

$$\lambda(\varepsilon)=\left[1-\frac{\varepsilon}{\varepsilon-\rho}\int_0^\infty dy\; e^{-y}\varphi_y(y,t)\right]=\frac{4}{3\pi}\,\frac{\theta^{3/2}\theta_t}{T-\theta},$$

$$\lambda(0)=1, \tag{46}$$

was used. Now Eq. (43) for $\varphi(y,t)$ can be rewritten in the form

$$\frac{2}{\rho\sqrt{\pi}}\,\hat{L}f+\varepsilon\,\frac{\partial\varphi}{\partial x}=(\varepsilon-\rho)\left[1-\frac{4}{3\pi}\,\lambda(\varepsilon)x^{3/2}\right]. \tag{47}$$

Certainly, the conditions (44) have to be satisfied. To find the exact solution of Eq. (47), we use the properties of the operator $\hat{L}$;

$$\frac{d}{dx}\,\frac{1}{\sqrt{x}}\,\frac{d}{dx}\,\hat{L}\phi=\frac{1}{D(x)}\,\frac{d}{dx}\,\frac{D^2(x)}{\sqrt{x}}\,\varphi'',$$

$$D(x)=\int_0^x dy\; y^{1/2}e^{-y}$$

$$\lim_{x\to\infty}\hat{L}\varphi\to D(\infty)\varphi'(x)=\frac{\sqrt{\pi}}{2}\,\varphi'(x). \tag{48}$$

As a result, for $\varphi(x,\varepsilon)$ and $\lambda(\varepsilon)$, we obtain a system of equations

$$\varphi(x,\varepsilon)=\left(\frac{\rho}{\varepsilon}-1\right)u(x,\varepsilon)=\left(\frac{\theta}{T}-1\right)u(x,\varepsilon),$$

$$\frac{d^2u}{dx^2}=\frac{\sqrt{\pi}}{2}\,\lambda(\varepsilon)\varepsilon(1+\varepsilon)x^{1/2}\left[D(x)+\frac{\sqrt{\pi}}{2}\,\varepsilon\right]^{-2},$$

$$\frac{du}{dx}\,(0,\varepsilon)=-1; \tag{49}$$

$$\int_0^\infty dy\; y^{1/2}e^{-y}u(y)=\int_0^\infty dy\; y^{3/2}e^{-y}u(y)=0. \tag{50}$$

It follows from the conditions (50) that

$$\int_0^\infty dy\; y^{3/2}e^{-y}u'_y(y)=0. \tag{51}$$

For our aim it is important to calculate precisely the function $\lambda(\varepsilon)$ (46), which defines the temperature equation for θ. We find the derivative of $u(x,\varepsilon)$,

$$u_x(x,\varepsilon)=-1+\frac{\sqrt{\pi}}{2}\,\lambda(\varepsilon)\varepsilon(1+\varepsilon)\times\int_0^x dy\; y^{1/2}\left[D(x)+\frac{\sqrt{\pi}}{2}\,\varepsilon\right]^{-2},$$

then substitute this expression into Eq. (51), and obtain the result

$$\lambda^{-1}(\varepsilon)=\frac{2}{3}\,\varepsilon(1+\varepsilon)\int_0^\infty dx\,x^{3/2}e^{-x}$$

$$\times\int_0^x dy\,y^{1/2}\left[D(x)+\frac{\sqrt{\pi}}{2}\,\varepsilon\right]^{-2}. \tag{52}$$

Let us find the behavior of the function $u(x,\varepsilon)$ under $x \ll 1$. In this case we set, in Eq. (52), $D(x)\simeq 2x^{3/2}/3$. With this condition the derivative of u becomes

$$u_x(x,\varepsilon)=-1+\frac{\sqrt{\pi}}{2}\,\Lambda(\varepsilon)\varepsilon(1+\varepsilon)\left[x^{3/2}\,\frac{2}{3}+\frac{\sqrt{\pi}}{2}\,\varepsilon\right]^{-1}$$

$$+\lambda(\varepsilon)(1+\varepsilon).$$

In the limiting case, as $\varepsilon\to 0$, we require that $u(x)\to 0$ as $x\to\infty$ because the function $u(x,\varepsilon)$ at $x\simeq 1$ has to have small magnitudes in comparison to its values in the boundary layer. This rough consideration leads to the expression

$$u(x,\varepsilon)=u(0,\varepsilon)-\frac{\sqrt{\pi}}{2}\,\varepsilon\int_0^x\frac{dy}{\left[\frac{2}{3}\,y^{3/2}+\frac{\sqrt{\pi}}{2}\,\varepsilon\right]}$$

$$=\int_x^\infty\frac{dy}{\left[\frac{4}{3\sqrt{\pi}}\,y^{3/2}+\varepsilon\right]}.$$

Here the condition $\lambda(0)=1$ was used.

Finally, in the boundary layer near $x=0$, the function $u(x)$ can be presented in the form

$$u(x)\simeq\delta\psi(x/\delta),\qquad \delta=\left(\frac{3\sqrt{\pi}}{4}\,\varepsilon\right),\qquad \psi(x)=\int_x^\infty\frac{dy}{[1+y^{3/2}]} \tag{53}$$

and the integral $\psi(x)$ can be expressed in elementary functions

$$\psi(x)=\frac{\pi}{\sqrt{3}}+\frac{1}{3}\ln\frac{(1+\sqrt{x})^2}{1-\sqrt{x}+x}-\frac{2}{\sqrt{3}}\arctan\frac{2\sqrt{x}-1}{\sqrt{3}},$$

$$\psi(0)=\frac{4\pi}{3\sqrt{3}}.$$

Thence the electron distribution is defined as

$$f(x,t)=\frac{2}{\sqrt{\pi}}\,e^{-x}\left\{1+\frac{\theta-T}{T}\left[\delta\psi\left(\frac{x}{\varepsilon}\right)+O(\varepsilon)\right]\right\}. \tag{54}$$

The rigorous investigation of the asymptotic quality of the constructed exact solution of Eqs. (46) and (47) for φ,λ shows that formula (53) is valid within the accuracy up to the terms of $O(\varepsilon)$. We shall be limited by two terms of the asymptotic expansion on the right-hand side of Eq. (52), as $\varepsilon\to 0$, and shall obtain the result

$$\lambda(\varepsilon)\approx[1+A\varepsilon^{2/3}]^{-1},\qquad A=\psi(0)\left(\frac{3\sqrt{\pi}}{4}\right)^{2/3}\approx 2.9. \tag{55}$$

On the other hand, the substitution of Eq. (53) into Eq. (46) leads to the expression

$$\lambda(\varepsilon)\approx 1-\delta\psi(0)=1-A\varepsilon^{2/3}, \tag{56}$$

which coincides with Eq. (55) for $\varepsilon\to 0$. The results of the numerical calculations that will be presented below show that it is better to use for $\lambda(\varepsilon)$ formula (55) in the temperature equation

$$\theta^{3/2}\theta_t=\frac{3\pi}{4}\,\frac{T-\theta}{1+2.9\varepsilon^{2/3}}.$$

VI. ION DISTRIBUTION FUNCTION

In this section the equations for the ion distribution function are considered in the thermal and the superthermal velocity regions. In particular, it is shown that the ion function deviation from equilibrium is 10^3 times smaller than the corresponding value for the electron distribution function for a time comparable with the temperature equalization time period ($t\simeq\tau_{\theta_0}\sim 1$).

According to the previous results, the equation for $F(x,t)$ ($\varepsilon\ll 1$) can be obtained from Eq. (21) by the substitution of the limiting ($\rho=0$) solution for the electron distribution function and temperatures. Otherwise, we take in Eq. (21)

$$f(0)=\int_0^\infty dy\,f(y)=\frac{2}{\sqrt{\pi}},$$

$$T^{1/2}\,\frac{dT}{dt}=-T^{1/2}\,\frac{d\theta}{dt}=-\frac{4}{3\sqrt{\pi}}\left(\frac{T}{\theta}\right)^{3/2}\left[1-\frac{\theta}{T}\right]$$

and obtain

$$T^{3/2}\,\frac{\partial F}{\partial t}=\frac{1}{\sqrt{x}}\,\frac{\partial}{\partial x}\left\{\frac{1}{\sqrt{\rho}}\,\hat{J}(F,F)+\frac{4}{3\sqrt{\pi}}\right.$$

$$\left.\times\sqrt{\frac{T}{\theta}}\,x^{3/2}\left(\frac{\partial F}{\partial x}+F\right)\right\}, \tag{57}$$

where $T(t)$ and $\theta(t)$ are known functions. As it follows from Eq. (57), in the thermal velocity region $x\ll\varepsilon^{-1/3}\ll 1$ a good approximation for $F(x,t)$ is the expression

$$F(x,t)\simeq F_1\left[x,\int_0^t\frac{dt'}{\rho^{1/2}T^{3/2}(t')}\right], \tag{58}$$

where $F_1(x,t)$ is the solution of the Cauchy problem

$$\frac{\partial F}{\partial\tau}=\hat{J}(F_1,F_1),\qquad F_1(x,0)=F_0(x) \tag{59}$$

[compare with the solution (15)]. Therefore, subjected to the inequality $\varepsilon \ll 1$, the electrons' influence on the ion distribution function actually leads to the changing of temperature $T(t)$ only and to the corresponding time variable replacement. The electrons' influence on the form of the function $F(x,t)$ is detected only for $x \gg \varepsilon^{-1/3} \gg 1$. In this region, using the notation of Eq. (58), the equation for $F_1(x,\tau)$ can be written as

$$\frac{\partial F}{\partial \tau} = \frac{1}{\sqrt{x}} \frac{\partial}{\partial x} \left[1 + \frac{4}{3\sqrt{\pi}} \sqrt{\varepsilon(\tau)} x^{3/2} \right] \left(\frac{\partial F}{\partial v} + F \right). \tag{60}$$

For instance, if initially ions have a monoenergetic distribution $F_0(x) = (\frac{2}{3})^{1/2} \delta(x - \frac{3}{2})$, then the electron action results in a more rapid propagation of $F_1(x,\tau)$ into the high-velocity region $x \gg 1$ than that described by Eq. (59). Actually, electrons accelerate the formation of the ion tail. For $t \gg \rho^{1/2}$ the deviation of the function $F(x,t)$ from the Maxwell distribution vanishes because the latter is an exact solution of Eq. (57). Based on the exact equations (10) and (11), it can be shown that the deviation of $F(x,t)$ from the equilibrium distribution is of $O(\varepsilon^{3/2})$ for $t \sim 1$. Roughly speaking, this means that this deviation is 10^3 times smaller than the corresponding value for the electron distribution function.

VII. RELAXATION PROCESS OF COLD ELECTRONS AND HOT IONS

In Sec. III the weakly nonisotermic plasma relaxation was described in the standard approach frames for $\varepsilon(0) \ll 10^{-2}$ $(\theta_0 \gg \rho^{1/3} T_0)$. In other sections the problem was examined in a different way and some different results were obtained. In particular, the validity of the usual description was extended up to $\varepsilon(0) \ll 1$. In practice, different situations occur when the last inequality at $t=0$ is not valid. An example is the neutral beam injection into a magnetic trap when originally the electron and ion thermal velocities are equal to $\theta_0 = \rho T_0$, that is, $\varepsilon(0) = 1$ and $T_0 \sim \rho^{-1} \theta_0$. In this section we emphasize distinctive features of the relaxation in possibly extreme temperature regimes. The first stage of the temperature equilibration can be separated in the initial stage, which occurs during the short time period $t_1 \ll \tau_m$, and the intermediate initial stage, which occurs during the time period compared with electron-electron collisional time $t_2 \simeq \tau_m$. In the first stage, the behavior of $f(x,t)$ and $\varepsilon(t)$ depends essentially on the initial conditions. In spite of this circumstance, the temperature evolution can be described by formulas (17) and (46), but the behavior of $\lambda(\varepsilon)$ for $\varepsilon \sim 1$ does not have a universal character. It is conducted by the initial distributions. At the end of the initial stage $\varepsilon(t)$ diminishes its value from ε_0 to $\varepsilon(t_1) = 0.1$–0.2. The intermediate stage has no analog in the usual picture of relaxation for the weakly nonisotermic plasma. The formulas obtained in Secs. IV and V are valid for the intermediate stage $(t_2 \ll \tau_M)$. The electron temperature rises rapidly and becomes 100 times greater than the initial value. At the same time, the energy spectrum of the ions is maintained very close to the initial distribution. Next comes the ion relaxation stage $(t_3 \sim \tau_M)$. The ion distribution takes a quasiequilibrium shape at the instant t_3 when the ion and electron temperatures differ only by a factor 3. The last stage of temperature equalization does not differ from the classical relaxation picture. Further on in the section the quantitative estimates of the process will be given when a comparison of the above-obtained asymptotic formulas with the numerical results will be presented. Typical initial conditions for numerical simulations are the Maxwell distributions or the monoenergetic distributions $f_0(x) = F_0(x) \sim \delta(x - x_0)$. The last initial distributions model the neutral beam injection conditions.

The initial stage of relaxation occurs during the time interval of order $t_1 \simeq \rho^{5/2}$. It is characterized by the changing of $\varepsilon(t)$ from $\varepsilon(0) = 1$ to the magnitudes $\varepsilon \ll 1$. The asymptotic equations from Sec. V are inapplicable and this very short stage can be described in another way.

We suppose that $\varepsilon(0) = \varepsilon_0$ is finite at the instant $t=0$ and we shall briefly describe a transit process that finally leads to the execution of the condition $\varepsilon \ll 1$. We return to the exact equation (10) and pass to the conditional limit, as $\rho \to 0$, under the two following conditions: the first is that the electron temperature tends to the initial one $\theta \to 0$ and θ/ρ is finite and the second is that at the initial time period $t \to 0$, $t\rho^{-5/2}$ is finite. In this case $T \to 2$, as $\rho \to 0$, and $\varepsilon \to 2\rho/\theta$ is finite. Under this limit the ion distribution function remains unchanged $F(x,t) \to F_0(x)$. Set $t' = t\rho^{-5/2}$. Then we get for the electron distribution function $f(x,t) = f'(x,t')$ the equation

$$\begin{aligned} \frac{2}{3} \frac{\partial f}{\partial t} = \frac{1}{\sqrt{x}} \frac{\partial}{\partial x} \Bigg\{ \hat{J}(f,f) + \frac{2}{3} \varepsilon \frac{\partial f}{\partial x} \Bigg[\int_0^{x/\varepsilon} dy\, y^{3/2} F_0(y) \\ + \left(\frac{x}{\varepsilon} \right)^{3/2} \int_{x/\varepsilon}^{\infty} dy\, F_0(y) \Bigg] \\ - \left(\frac{2}{3} \right)^{3/2} \varepsilon^{-1} \frac{d\varepsilon}{dt} x^{3/2} f \Bigg\}, \quad f|_{t=0} = 0. \end{aligned} \tag{61}$$

As earlier, the prime is dropped in the final result. The equation for $\varepsilon(t)$ is obtained from the normalization condition

$$\frac{d\varepsilon}{dt} = -\frac{1}{3\sqrt{2}} \varepsilon^{7/2} \int_0^{\infty} dx\, F_0(x) \int_0^x dy\, y^{1/2} f(\varepsilon y),$$

$$\varepsilon|_{t=0} = \varepsilon_0 .$$

Let us recall that the preceding equation takes the place of the temperature equation for $\theta(t)$ because in the conditional limit case under consideration $\theta = 2\rho/\varepsilon$ and ε is finite, as $\rho \to 0$. Using notation similar to that in Eq. (46), the preceding equation can be rewritten as

$$\begin{gathered} \frac{d\varepsilon}{dt} = -\frac{4}{3\sqrt{\pi}} \lambda(\varepsilon) \varepsilon^{7/2}, \\ \lambda(\varepsilon) = \frac{\sqrt{\pi}}{2} \int_0^{\infty} dx\, F_0(x) \int_0^x dy\, y^{1/2} f(\varepsilon y). \end{gathered} \tag{62}$$

From here it follows that the equation for $\theta(t)$, where t is the original variable, for all values of ε can be presented approximately in the form

$$\theta^{3/2}\frac{d\theta}{dt}=\frac{4}{3\sqrt{\pi}}\lambda(\varepsilon)(T-\theta),\quad \varepsilon=\rho\frac{T}{\theta},$$

where $\lambda(\varepsilon)$ for $\varepsilon\ll 1$ is obtained in Sec. V. For $\varepsilon\gg 1$ the value of $\lambda(\varepsilon)$ has to be defined from the solutions of Eqs. (61) and (62). The dependence of $\lambda(\varepsilon)$ on t for $\varepsilon\sim 1$ depends essentially on the initial conditions $f_0(x)$ and $F_0(x)$. For $\lambda(\varepsilon)$ under $\varepsilon\gg 1$, we have the expression

$$\lambda(\varepsilon)\simeq\varepsilon^{-3/2}\frac{\sqrt{\pi}}{2}\int_0^\infty dx\, F_0(x),$$

which, for the Maxwell distribution $F_0(x)$, coincides with the known [10] value

$$\lambda_s(\varepsilon)=(1+\varepsilon)^{-3/2},\quad \lim_{\varepsilon\to\infty}\lambda_s(\varepsilon)\to\varepsilon^{-3/2}.$$

Simultaneously with the decreasing of $\varepsilon(t)$ over the interval $t_1\sim\rho^{5/2}$, the electron distribution function assumes its quasiequilibrium shape. The deviation of $F(x,t)$ and $T(t)$ from the initial conditions are negligible. The numerical computations show that the dependence of $\lambda(\varepsilon)$ and $f(x,t)$ on the initial data is already masked at $\varepsilon\simeq 0.1$–0.2, i.e., when the electron temperature rises 5–10 times.

The intermediate stage of the temperature equalization occurs over the interval $\rho^{5/2}\ll t\ll\rho^{1/2}$. In this stage the electron function $f(x,t)$ keeps its form and depends on time only through $\varepsilon(t)\ll 1$,

$$f(x,t)\simeq\frac{2}{\sqrt{\pi}}e^{-x}[1-\varphi+O(\varepsilon)],\quad \varphi=\left[\frac{T-\theta}{T}\,\delta\psi\left(\frac{x}{\delta}\right)\right].$$

The electron function $f(x,t)$ is near the equilibrium distribution and changes very slowly in time. Temperature equilibration is practically subjected to the common law. The temperatures $T(t)$ and $\theta(t)$ satisfy the equations

$$T+\theta=2,$$

$$\theta^{3/2}\frac{d\theta}{dt}=\frac{4}{3\sqrt{\pi}}\lambda_a(\varepsilon)(T-\theta),\tag{63}$$

$$\lambda_a(\varepsilon)=\frac{1}{1+2.9\varepsilon^{2/3}},$$

where $\lambda_a(\varepsilon)$ is obtained in Sec. V. Usually the electron temperature $\theta(t)$ rises very rapidly and the ion temperature $T(t)$ decreases very slowly, while the ion function does not modify its shape $F(x,t)\sim F_0(x)$ and can be very far from the equilibrium state. In accordance with the results of Sec. VI, the function $F(x,t)$ at $\varepsilon\ll 1$ has the form

$$F(x,t)=F_1\left[x,\int_0^t dt'\rho^{-1/2}T^{-3/2}(t')\right],\tag{64}$$

where $T(t)$ satisfies Eq. (63) and $F_1(x,\tau)$ is a solution of the Cauchy problem for the Landau-Fokker-Planck equation

$$\frac{\partial F_1}{\partial\tau}=\frac{1}{\sqrt{x}}\frac{\partial}{\partial x}\hat{J}(F_1,F_1),\quad F_1(x,0)=F_0(x).\tag{65}$$

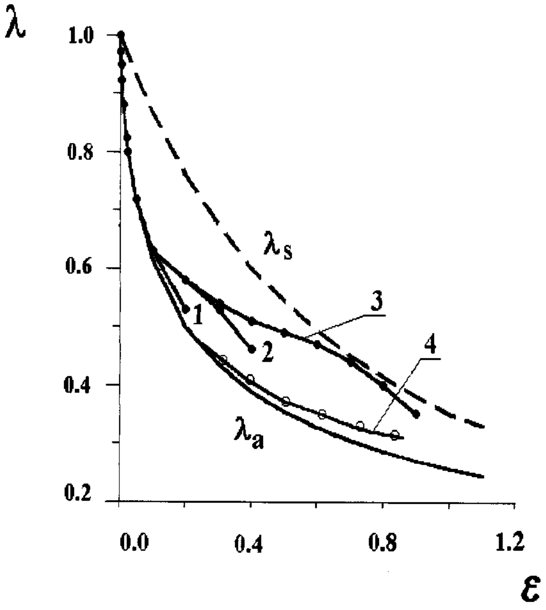

FIG. 2. Dependence of the function $\lambda(\varepsilon)$ on ε obtained analytically: λ_s (dashed line) in [10] and λ_a (solid line) in the present work. The numerical results for λ_n are presented for the initial δ functions (black circles), where $1-\varepsilon_0=1.0$, $2-\varepsilon_0=0.5$, and $3-\varepsilon_0=0.25$, and 4 (open circles) for the initial Maxwell functions, where $\varepsilon_0=1.0$.

In this equation we discard the term proportional to ε [compare with Eq. (57)], which is important only in the far tail region $x\gg\varepsilon^{-1/3}$. In this stage it is still possible to suppose the ion temperature to be close to the initial one $T(t)\approx T(0)\simeq 2$. Whence, recalling Eqs. (64) and (65), it follows that the end of this stage is defined by the interval $t_2\ll 2^{3/2}\rho^{1/2}$. To evaluate the variation of $\theta(t)$ and $\varepsilon(t)$ over this time period we take into account that in Eq. (63) $\theta\ll T\simeq 2$ and $\varepsilon\ll 1$. Therefore, we obtain the approximate solution for the time intervals $t\gg\rho^{5/2}$,

$$\theta(t)\simeq\left[\frac{20}{3\sqrt{\pi}}t\right]^{2/5}\left\{1+\frac{3\sqrt{\pi}}{20}\frac{\theta^{5/2}}{t}\right\}^{2/5}\approx\left[\frac{20}{3\sqrt{\pi}}t\right]^{2/5}\approx 2t^{2/5},$$

where the infinitesimal value of $\theta_0=O(\rho)$ is taken into account. From the preceding equation it can be seen that $\theta(t_2)\ll\rho^{1/5}$ and $\varepsilon(t_2)\gg 2\rho^{4/5}$. Quantitatively, for the real mass ratio $\rho\leqslant 10^{-3}$ at the instant t_2, when the function $F(x,t)$ begins to change, the inequalities $\theta(t_2)\ll 0.25$ and $\varepsilon(t_2)\gg 0.01$ are valid. Roughly speaking, this means that the electron temperature rises 100 times in comparison to the initial magnitude before the ion function starts to change noticeably. This consideration and the asymptotic formulas are confirmed by the numerical calculations for $\varepsilon\leqslant 0.2$ (Fig. 2).

The ion Maxwellization stage has the duration $t_3\simeq\rho^{1/2}\gg t_2$, which is small compared to the temperature equilibration time. The relaxation process is described by Eqs. (64) and (65). The striking feature of this stage is that the ion function $F(x,t)$ changes simultaneously with the temperature equalization (63). The final moment of the stage can be assumed at the moment $t_3 \approx 2^{3/2}\sqrt{\rho}$, after which the function $F(x,t)$ maintains the Maxwell form. The estimations show that at the end of this stage $\varepsilon(t_3) \sim \rho^{1/5}$. From the numerical results it is obtained that for the real electron-ion mass ratio the temperatures of the plasma components differ only three times at the instant t_3, $T/\theta \simeq 3$. Moreover, the value of the temperature ratio at the instant t_3 depends on the mass ratio: for $\rho \simeq 10^{-2}$, 10^{-3}, and 6.25×10^{-4} the corresponding values of T/θ are equal to 2.1, 2.85, and 3.1.

The final stage of the temperature equilibration does not differ from the similar stage that has already been described in Sec. III, except for the corrections in the temperature equation. The values of these corrections are several percent of the mass ratio and do not play a significant role in the solution of the initial-value problem. However, they can be sufficient in the consideration of stationary problems. The existence of such corrections is in good agreement with the numerical results (Fig. 2).

VIII. CONCLUSION

The temperature relaxation process of a space uniform plasma composed of electrons and one species of ions is considered for various initial temperatures. A detailed description of this classical problem of the collisional plasma kinetic theory is given.

The classical relaxation picture is characterized by a weakly nonisotermic situation $\rho^{1/3} \ll T_0/\theta_0 \ll \rho^{-1/3}$; otherwise $\varepsilon_0 \ll 10^{-2}$. To study the problem a standard approach is usually used, when the electron and ion distribution functions are supposed to be a Maxwellian with time-dependent electron $T_e(t)$ and ion $T_i(t)$ temperatures. Here we show that, although the general concepts are correct, many limiting problems remain.

A deeper examination of the problem shows that the usual perturbation theory cannot be used. We show that the part of the perturbation of the electron distribution has the character of a boundary layer in the neighborhood of small velocities. In this work the boundary layer is thoroughly studied. The asymptotic behavior of the solution, as $\rho = m_e/m_i \ll 1$, is considered for the arbitrary initial electron and ion distribution functions. Previous results are revised and corrected and alternative analytical formulas are obtained. The objective is to focus on a deviation from the classical relaxation picture. The obtained results can be used as a test in both analytical and numerical models of the collisional plasma.

In our study we consider in details the case in which the ratio of the initial ion-to-electron speeds is small, that is when for all $t \geqslant 0$ the inequality (32)

$$\varepsilon(t) = \frac{m_e}{m_i}\frac{T(t)}{\theta(t)} \ll 1$$

is valid.

There also exist deviations from the limiting ($\rho \ll 1$) solutions, which have of small corrections. These corrections play a significant role for the light- and heavy-ion relaxation. In spite of their smallness, the corrections are also interesting for the case under consideration since they appear due to a highly specific mechanism of heavy- and light-particle interactions. Then, as a results of the analysis of the exact equation (10) for the electron distribution function, in the vicinity of small velocities $x \sim 0$, the presence of a composed boundary layer $0 \leqslant x \leqslant \varepsilon$ and $\varepsilon \ll x \ll \varepsilon^{2/3}$ was established.

The corrected (under $\varepsilon \ll 1$) distribution electron function can be written in the form

$$f(x,t) = \frac{2}{\sqrt{\pi}} e^{-x}(1+\phi_a),$$

$$\phi_a = \frac{\theta - T}{T}\left[\delta(t)\psi\left(\frac{x}{\varepsilon}\right) + O(\varepsilon)\right],$$

$$\delta(t) = \left[\frac{3\sqrt{\pi}}{4}\varepsilon(t)\right]^{2/3},$$

where the function ψ is defined in Eq. (54). In the thermal velocity region, the maximum value of the deviation of the function $f(x,t)$ from the Maxwell distribution is located at zero velocity $x=0$:

$$f(0,t) \simeq \frac{2}{\sqrt{\pi}}(1+\phi_a), \quad \phi_a = 2.9\varepsilon^{2/3}\frac{\theta - T}{T}.$$

It should be remarked that the relative deviation of the electron distribution from equilibrium can be larger in the tail region ($x \to \infty$). However large it might be, the tail region contains an exponentially small amount of particles and it has no practical interest for the temperature relaxation problem. The comparison of the function ϕ_a with the function ϕ_n obtained from numerical calculations is presented below:

ε	0.030	0.008	0.005
ϕ_a/ϕ_n	0.280/0.230	0.100/0.100	0.070/0.074

The deviation of the ion distribution function $F(x,t)$ from equilibrium is of $O(\varepsilon^{3/2})$ for times long compared to the temperature equilibration time $t \sim 1$. It is interesting that the ion function deviation is 10^3 times smaller than the corresponding value for the electron distribution function.

The real condition for the applicability of the known formula for temperatures (17)

$$\theta^{3/2}\frac{d\theta}{dt} = \frac{4}{3\sqrt{\pi}}(T-\theta)$$

is the requirement (32) $\varepsilon \ll 1$. For the hydrogen plasma initially consisting of cold electrons and hot ions, the inequality (19) is fulfilled when the initial temperatures' ratio is on the order of 2–3. The requirement $\varepsilon \ll 1$ allows us to consider a plasma with initial temperatures that may have wider differences (about 100 times). This circumstance can be important for applications both in laboratory and in space plasmas.

The corrected formula for temperatures has the form

$$\theta^{3/2}\frac{d\theta}{dt} = \frac{4}{3\sqrt{\pi}}\lambda_a(\varepsilon)(T-\theta), \quad \lambda_a(\varepsilon) \simeq \frac{1}{1+2.9\varepsilon^{2/3}}$$

and can be used with high accuracy for $\varepsilon \leqslant 0.2$. These corrections may play a significant role when stationary problems are considered.

In Sec. VII the relaxation of cold electrons and hot ions was considered in the possibly extreme temperature regimes $(\varepsilon \simeq 1)$. The obtained asymptotic formulas are in good agreement with the numerical simulation results.

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